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Third Order Efficient Tests in Exponential Lattice Models

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In full rank multivariate exponential families of lattice distributions, one sided testing problems are considered. For these testing problems the uniformly most powerful unbiased tests are not uniformly efficient of order $o(n^{-1})$ in the class $\mathscr{T}_2^*$ of all tests which are asymptotically similar of order $o(n^{-1})$ (Hipp, C. (1983). *J. Multivariate Anal.* **13** 67–108). In this note, better asymptotically similar tests are constructed which—for one specified value of the nuisance parameter—have maximal power in the class $\mathscr{T}_2^*$. The power of tests is computed up to terms of order $o(n^{-1})$, at contiguous alternatives, and for the specified value of the nuisance parameter. © 1986 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

Let p be a positive integer and v a probability measure on the Borel field $\mathscr{B}^{p+1}$ of $\mathbb{R}^{p+1}$ which is a lattice distribution with minimal lattice $\mathbb{Z}^{p+1}$. Assume that the natural parameter space

$$H^* = \left\{\eta \in \mathbb{R}^{p+1}: \int \exp(\eta^T z)\, v(dz) < \infty\right\}$$

has nonvoid interior H. For $\eta \in H$ define

$$f(\eta) = \log \int \exp(\eta^T z)\, v(dz)$$

and P_η as the probability measure on $\mathscr{B}^{p+1}$ with v-density

$$z \to \exp(\eta^T z - f(\eta)), \qquad z \in \mathbb{R}^{p+1}.$$

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Write θ for the first component of the $(p+1)$-vector $\eta \in H$. Fix $\theta_0 \in \mathbb{R}$ such that $T = \{\tau \in \mathbb{R}^p \colon (\theta_0, \tau) \in H\}$ is nonvoid.

For $n = 1, 2, \ldots$ we consider the one sided testing problem

$$\{P^n_\eta \colon \eta \in H, \theta = \theta_0\} \text{ against } \{P^n_\eta \colon \eta \in H, \theta > \theta_0\}. \tag{1.1}$$

Fix a level $\alpha \in (0, 1)$. For $\tau \in T$ let $\mathscr{T}^*_2(\tau)$ denote the class of all test sequences (φ_n) with the following property: Uniformly for $u \in \mathbb{R}^p$, $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau + u)} \varphi_n = \alpha + o(n^{-1}) \tag{1.2}$$

and write $\mathscr{T}^*_2$ for the class of all test sequences (φ_n) which are elements of $\mathscr{T}^*_2(\tau)$ for all $\tau \in T$.

For $\tau \in T$ write $\Sigma(\theta_0, \tau)$ for the covariance matrix of $P_{(\theta_0, \tau)}$. Partition this matrix as follows:

$$\Sigma(\theta_0, \tau) = \begin{pmatrix} \sigma_{00}(\theta_0, \tau) & \Sigma_{01}(\theta_0, \tau) \\ \Sigma_{10}(\theta_0, \tau) & \Sigma_{11}(\theta_0, \tau) \end{pmatrix}$$

where $\Sigma_{11}(\theta_0, \tau)$ is a (p, p)-matrix. Let

$$M(\theta_0, \tau) = \Sigma_{01}(\theta_0, \tau)\, \Sigma_{11}^{-1}(\theta_0, \tau)$$

and $T_0 = \{\tau \in T \colon M(\theta_0, \tau) \notin Q^p\}$.

For $n = 1, 2, \ldots$ let φ^0_n be the uniformly most powerful unbiased level α test for (1.1) (see [3, pp. 134, 135]). Then $(\varphi^0_n) \in \mathscr{T}^*_2$. It is the purpose of this note to prove the following:

THEOREM. (i) *For every $\tau_0 \in T_0$ there exists $(\varphi^*_n) \in \mathscr{T}^*_2$ with the following properties.*

(a) *For all $(\varphi_n) \in \mathscr{T}^*_2(\tau_0)$ and all $t > 0$*

$$\limsup_n n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} (\varphi_n - \varphi^*_n) \leqslant 0.$$

(b) *For all $\tau \in T$ and $t > 0$*

$$\limsup_n n P^n_{(\theta_0 + tn^{-1/2}, \tau)} (\varphi^0_n - \varphi^*_n) \leqslant 0.$$

(c) *For all $t > 0$*

$$\limsup_n n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} (\varphi^0_n - \varphi^*_n) < 0.$$

(ii) *There exists a sequence of tests* (φ_n^{**}) *with the following property: If* $\tau_0 \in T_0$, *then* $(\varphi_n^{**}) \in \mathscr{T}_2^*(\tau_0)$ *and for all* $t > 0$

$$\limsup_n n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)}(\varphi_n^0 - \varphi_n^{**}) < 0.$$

Remark 1. Properties (b) and (c) of (φ_n^*) yield that (φ_n^0) is—up to terms of order $o(n^{-1})$—not asymptotically admissible in $\mathscr{T}_2^*$ if $T_0 \neq \varnothing$.

This is true in Example 3 of [2, p. 74]. In Theorem 4 of [2, p. 81] inadmissibility of (φ_n^0) was stated under restrictive assumptions on v and α.

Remark 2. Theorem 6 of [2, p. 82] states that for all $t > 0$ and $(\varphi_n) \in \mathscr{T}_2^*$

$$P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi_n \leqslant H_{2n}^{\alpha}(t, \theta_0, \tau_0) + o(n^{-1})$$

whenever $\tau_0 \in T_0$. Here, $H_{2n}^{\alpha}(t, \theta_0, \tau_0)$ is the envelope power function for $\mathscr{T}_2^*$ in the continuous case (see [4]). In our proof we shall derive the relation

$$P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi_n^* = H_{2n}^{\alpha}(t, \theta_0, \tau_0) + o(n^{-1}).$$

Hence $H_{2n}^{\alpha}(t, \theta_0, \tau_0)$ is an envelope power function for $\mathscr{T}_2^*$ at τ_0 in the lattice case, too, provided $\tau_0 \in T_0$.

It was stated in [2, p. 72] that "randomization diminishes power." In the construction of (φ_n^*) and (φ_n^{**}) we change the randomization of (φ_n^0) appropriately. To explain this more precisely we need some notations. Let $Z_1, Z_2, \ldots$ be independent $(p+1)$-dimensional random vectors with distribution $P_{(\theta,\tau)}$. For $n = 1, 2, \ldots$ write $Z_n = (X_n, Y_n)$ with univariate X_n and p-variate Y_n, and denote $X = X_1 + \cdots + X_n$ and $Y = Y_1 + \cdots + Y_n$. The tests (φ_n^0) can be written

$$\begin{aligned} \varphi_n^0 &= 1 && \text{if } X > k_n(Y, \alpha) \\ &= \gamma_n(Y, \alpha) && \text{if } X = k_n(Y, \alpha) \\ &= 0 && \text{if } X < k_n(Y, \alpha) \end{aligned}$$

with $k_n(Y, \alpha) \in \mathbb{Z}$ and $\gamma_n(Y, \alpha) \in [0, 1)$. The tests φ_n^{**} are defined as follows:

$$\begin{aligned} \varphi_n^{**} &= 1 && \text{if } X > k_n(Y, \alpha) \\ &= h(\gamma_n(Y, \alpha)) && \text{if } X = k_n(Y, \alpha) \\ &= 0 && \text{if } X < k_n(Y, \alpha). \end{aligned}$$

Here, h: $[0, 1] \to [0, 1]$ is a smooth function satisfying $h(0) = h(1) = 0$ and $\int_0^1 h(x)\, dx = 0$, and $h(x) < x$ if $x < \frac{1}{2}$, $h(x) > x$ if $x > \frac{1}{2}$.

The proof for $(\varphi_n^{**}) \in \mathscr{T}_2^*$ is based on the fact that $\gamma_n(Y, \alpha)$ is asymptotically uniformly distributed under $P^n_{(\theta_0, \tau_0)}$ whenever $\tau_0 \in T_0$.

The test sequence (φ_n^*) is constructed as follows: First we construct a sequence (φ_n') as above with smooth functions h_n which vary with n such that $h_n(x) \to 0$ if $x < \frac{1}{2}$, $h_n(x) \to 1$ if $x > \frac{1}{2}$.

The sequence (φ_n') is in $\mathscr{T}_2^*(\tau_0)$ and satisfies

$$P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi_n' = H^\alpha_{2n}(t, \theta_0, \tau_0) + o(n^{-1}).$$

We take a $\sqrt{n}$-consistent estimator τ_n for τ and define φ_n^* as φ_n' if τ_n is close to τ_0, and as φ_n^0 elsewhere.

Remark 3. If $M(\theta_0, \tau_0) \in Q^p$ then the sequence (φ_n^{**}) constructed in the proof for part (ii) of the theorem will not satisfy $(\varphi_n^{**}) \in \mathscr{T}_2^*(\tau_0)$ *and*

$$\limsup_n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)}(\varphi_n^0 - \varphi_n^{**}) < 0 \qquad \text{for all } t > 0$$

in general. In Example 2 of [2, p. 73] the sequence (φ_n^0) satisfies

$$\liminf_n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)}(\varphi_n^0 - \varphi_n) \geqslant 0$$

for all $t > 0$ and $(\varphi_n) \in \mathscr{T}_2^*(\tau_0)$ if $\alpha = \frac{1}{2}$, and

$$\limsup_n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)}(\varphi_n^0 - \varphi_n) \geqslant 0$$

for all $t > 0$ and $(\varphi_n) \in \mathscr{T}_2^*(\tau_0)$ for arbitrary α.

Remark 4. Let $M(\theta_0, \tau_0) = (1/m)(k_1, ..., k_p)$ with relatively prime integers $m, k_1, ..., k_p$. Then the distribution of $\gamma_n(Y, \alpha)$ can be approximated by $\delta(a_n) * U * V$, where $a_n \in \mathbb{R}$, $\delta(a_n)$ is the Dirac measure at a_n, U is the uniform distribution on $\{0, 1/m, ..., (m-1)/m\}$, and V is a noncentral chi-square distribution; the convolution is computed on the torus $\mathbb{R}/\mathbb{Z}$. The constants a_n oscillate, and the oscillation depends on the value of the nuisance parameter τ_0. A function h yielding a sequence (φ_n^{**}) satisfying $(\varphi_n^{**}) \in \mathscr{T}_2^*(\tau_0)$ has to satisfy the conditions $Eh^*(U+x) = EUh^{*\prime}(U+x) = 0$ for all x, where $h^*(x) = h(x) - x$. These conditions, however, imply $h^* \equiv 0$.

2. Proof of the Theorem

In this section we shall use some notations of [2] without repeating the definitions.

For $j=1, 2,\ldots$ let $h_j^*\colon [0,1]\to[0,1]$ be a trigonometric polynomial of the form

$$h_j^*(x)=\sum\{a_l\exp(2\pi i l x)\colon 0<|l|\leqslant j\}$$

with $h_j^*(0)=0$.

Assume that $h_j^*(x)<0$ for $0<x<\frac{1}{2}$, $h_j^*(x)>0$ for $\frac{1}{2}<x<1$, and that for all $0<\varepsilon<\frac{1}{2}$, $h_j^*(x)\to -x$, uniformly for $\varepsilon<x<\frac{1}{2}-\varepsilon$, and $h_j^*(x)\to 1-x$, uniformly for $\frac{1}{2}+\varepsilon<x<1-\varepsilon$. Let $h_j(x)=h_j^*(x)+x$. For $n=2, 3,\ldots$ let $\tau_n(Z_1,\ldots,Z_n)$ be an estimator for τ such that uniformly for (θ,τ) in compact subsets of H

$$P^n_{(\theta,\tau)}\{|\tau_n(Z_1,\ldots,Z_n)-\tau|\geqslant n^{-1/2}\log n\}=o(n^{-1}).$$

To simplify our notation we shall omit arguments whenever possible. Our first result yields a stochastic expansion for $\gamma_n(Y,\alpha)$.

Lemma 1. *We have uniformly for τ in compact subsets of T and $y\in A(n,(\theta_0,\tau))$*

$$\begin{aligned}\gamma_n(y,\alpha)&=c+Q_1(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)\\&\quad+n^{-1/2}\{-Q_2(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)\\&\quad+Q_1(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)\,W_1(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)\\&\quad+\tfrac{1}{2}\sigma^{-1}N_\alpha\gamma_n^2(y,\alpha)\\&\quad-\sigma^{-1}N_\alpha\gamma_n(y,\alpha)\,Q_1(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)\}+o(n^{-1/2}).\end{aligned}$$

Recall that

$$c=c(y)=k_n(y,\alpha)-n\mu_0+nM\mu_1+n^{1/2}\sigma N_\alpha-My$$

and

$$v=v(y)=-\sigma N_\alpha+M\tilde{y}(n,\theta_0,\tau),$$

and that $Q_1(v,\tilde{y}(n,\theta_0,\tau),\theta_0,\tau)$ is a polynomial in $\tilde{y}(n,\theta_0,\tau)$ of degree two or less.

Lemma 2. *For $\tau\in T$ let $P(y,\tau)$ be a polynomial in $y\in\mathbb{R}^p$ such that the degree and the coefficients of P remain bounded as long as τ varies in compact subsets of T. Then for all $\tau\in T_0$ and $m=1, 2,\ldots$ we have uniformly for $|u|\leqslant n^{-1/2}\log n$*

$$\begin{aligned}P^n_{(\theta_0,\tau+u)}&[h_m^*(c(Y)+Q_1(v(Y),\tilde{Y}(n,\theta_0,\tau+u),\theta_0,\tau+u))\\&\times P(\tilde{Y}(n,\theta_0,\tau+u),\tau+u)]=o(n^{-1/2}).\end{aligned}$$

Proof. It suffices to prove the assertion for

$$h_m^*(x) = \exp(2\pi i m x), \qquad x \in \mathbb{R}, m \neq 0.$$

For $w \in \mathbb{R}^p$ consider

$$f(w) = P^n_{(\theta_0, \tau+u)} h_m^*(c(Y) + w^T \tilde{Y}(n, \theta_0, \tau+u)).$$

With

$$\begin{aligned} H(n) &= n^{1/2}\sigma(\theta_0, \tau+u)\, N_\alpha - n\mu_0(\theta_0, \tau+u) \\ &\quad + nM(\theta_0, \tau+u)\, \mu_1(\theta_0, \tau+u) \end{aligned}$$

we obtain

$$\begin{aligned} f(w) &= P^n_{(\theta_0, \tau+u)} h_m^*((-M(\theta_0, \tau+u) + n^{-1/2}w)^T\, Y \\ &\quad + H(n) - n^{1/2} w^T \mu_1(\theta_0, \tau+u)) \\ &= (P_{(\theta_0, \tau+u)} h_m^*((-M(\theta_0, \tau+u) + n^{-1/2}w)^T\, Y_1))^n \\ &\quad \times h_m^*(H(n) - n^{1/2} w^T \mu_1(\theta_0, \tau+u)). \end{aligned}$$

There exists a positive ε such that for $|u| \leqslant \varepsilon$ and $|W| \leqslant \varepsilon$ we have

$$|P_{(\theta_0, \tau+u)} h_m^*((-M(\theta_0, \tau+u) + W)^T\, Y_1)| \leqslant 1 - \varepsilon.$$

This implies that for all $s > 0$ and uniformly for $|u| \leqslant n^{-1/2} \log n$ and $|w| \leqslant \varepsilon n^{1/2}$

$$f(w) = o(n^{-s}). \tag{2.1}$$

The same is true for derivatives of all orders:

If $\beta_1, ..., \beta_p$ are nonnegative integers, then for all $s > 0$ and uniformly for $|u| \leqslant n^{-1/2} \log n$ and $|w| \leqslant \varepsilon n^{1/2}$

$$\frac{\partial^{\beta_1 + \cdots + \beta_p}}{\partial w_1^{\beta_1} \cdots w_p^{\beta_p}} f(w) = o(n^{-s}).$$

Let $K | \mathscr{B}^p$ be a probability measure satisfying $\int |x|^2\, K(dx) < \infty$ which has a characteristic function $\hat{K}(t)$ vanishing for $|t| > 1$. If g is a measurable function which is bounded by some polynomial, then (2.1) and (2.2) together with Lemma 11.6 in [1, p. 98] imply that for all $\tau \in T$ and $s > 0$ and uniformly for $|u| \leqslant n^{-1/2} \log n$

$$\int P^n_{(\theta_0, \tau+u)} h_m^*(c(Y))\, g(\tilde{Y}(n, \theta_0, \tau+u) + \varepsilon^{-1} n^{-1/2} w)\, K(dw) = o(n^{-s}).$$

Let $G\colon \mathbb{R}^p \to \mathbb{R}$ be the polynomial of degree two or less such that $G(\tilde{y}(n, \theta_0, \tau+u)) = Q_1(v(y), \tilde{y}(n, \theta_0, \tau+u), \theta_0, \tau+u)$, and let $g_m(y) = h_m^*(G(y))\, P(y, \tau+u)$. Then for all nonnegative integers $\beta_1, ..., \beta_p$ there exists a polynomial Q in y such that for all $u \in \mathbb{R}^p$ with $|u| \leqslant n^{-1/2} \log n$

$$\left| \frac{\partial^{\beta_1 + \cdots + \beta_p}}{\partial y_1^{\beta_1} \cdots \partial y_p^{\beta_p}} g_m(y) \right| \leqslant k^{\beta_1 + \cdots + \beta_p} Q(y).$$

This implies that for all $s > 0$ and nonnegative integers $\beta_1, ..., \beta_p$ we have uniformly for $|u| \leqslant n^{-1/2} \log n$

$$\int P^n_{(\theta_0, \tau+u)} h_m^*(c(Y)) \frac{\partial^{\beta_1 + \cdots + \beta_p}}{\partial y_1^{\beta_1} \cdots \partial y_p^{\beta_p}} g_m(y) \Big|_{y = \tilde{Y}(n, \theta_0, \tau+u) + \varepsilon^{-1} n^{-1/2} w} K(dw)$$
$$= o(n^{-s}).$$

The remainder $R_n(x, w)$ in the Taylor expansion

$$g_m(x) = \sum_{0 \leqslant \beta_1 + \cdots + \beta_p \leqslant 1} (-\varepsilon n^{1/2})^{-(\beta_1 + \cdots + \beta_p)} w_1^{\beta_1} \cdots w_p^{\beta_p}$$
$$\times \frac{\partial^{\beta_1 + \cdots + \beta_p}}{\partial y_1^{\beta_1} \cdots \partial y_p^{\beta_p}} g_m(y) \Big|_{y = x + \varepsilon^{-1} n^{-1/2} w} \prod_1^p (\beta_j!)^{-1} + R_n(x, w)$$

satisfies

$$R_n(x, w) \leqslant n^{-1} |w|^2 R(x),$$

where R is a polynomial which may depend on τ but not on u. Integrating both sides of the Taylor expansion with respect to $P^n_{(\theta_0, \tau+u)}$ and K we obtain that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau+u)} h_m^*(c(Y))\, g_m(\tilde{Y}(n, \theta_0, \tau+u)) = o(n^{-1/2}).$$

This implies the assertion of Lemma 2.

For fixed m define the sequence of tests (φ_n) by

$$\begin{aligned} \varphi_n &= 1 && \text{if} \quad X > k_n(Y, \alpha) \\ &= h_m(\gamma_n(Y, \alpha)) && \text{if} \quad X = k_n(Y, \alpha) \\ &= 0 && \text{if} \quad X < k_n(Y, \alpha). \end{aligned}$$

LEMMA 3. *For $\tau_0 \in T_0$ we have $(\varphi_n) \in \mathscr{T}_2^*(\tau_0)$.*

Proof. We have to show that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0+u)} h_m^*(\gamma_n(Y, \alpha))\, 1_{\{X = k_n(Y, \alpha)\}} = o(n^{-1}).$$

Recall that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0+u)}\{Y \notin A(n, (\theta_0, \tau_0+u))\} = o(n^{-1}).$$

With Lemma 1 we obtain that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$\begin{aligned}
&P^n_{(\theta_0, \tau_0+u)} h^*_m(\gamma_n(Y, \alpha))\, 1_{\{X = k_n(Y,\alpha)\}} \\
&\quad = n^{-1/2}\sigma^{-1}(\theta_0, \tau_0+u)\, \varphi(N_\alpha)\, P^n_{(\theta_0, \tau_0+u)}[\{1 + n^{-1/2}\sigma^{-1} N_\alpha \gamma_n(Y, \alpha) \\
&\qquad - n^{-1/2}\sigma^{-1} N_\alpha Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0) \\
&\qquad + n^{-1/2} W_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \theta_0, \tau_0)\} \\
&\qquad \times \{h^*_m(c(Y) + Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0+u)) \\
&\qquad + h^{*\prime}_m(c(Y) + Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0)) \\
&\qquad \times n^{-1/2}[-Q_2(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \theta_0, \tau_0) \\
&\qquad + Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0)\, W_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \theta_0, \tau_0) \\
&\qquad + \tfrac{1}{2}\sigma^{-1}(\theta_0, \tau_0)\, N_\alpha \gamma_n^2(Y, \alpha) \\
&\qquad - \sigma^{-1}(\theta_0, \tau_0)\, N_\alpha \gamma_n(Y, \alpha)\, Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0)]\}] \\
&\qquad + o(n^{-1}).
\end{aligned}$$

From Lemma 2 we obtain that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$\begin{aligned}
&P^n_{(\theta_0, \tau_0+u)} h^*_m(\gamma_n(Y, \alpha))\, 1_{\{X = k_n(Y,\alpha)\}} \\
&\quad = n^{-1}\sigma^{-1}(\theta_0, \tau_0+u)\, \varphi(N_\alpha)\, P^n_{(\theta_0, \tau_0+u)}[\sigma^{-1} N_\alpha \gamma_n(Y, \alpha)\, h^*_m(\gamma_n(Y, \alpha)) \\
&\qquad + \tfrac{1}{2}\sigma^{-1} N_\alpha \gamma_n^2(Y, \alpha)\, h^{*\prime}_m(\gamma_n(Y, \alpha)) \\
&\qquad - \sigma^{-1} N_\alpha \gamma_n(Y, \alpha)\, Q_1(v(Y), \tilde{Y}(n, \theta_0, \tau_0+u), \tau_0)\, h^{*\prime}_m(\gamma_n(Y, \alpha))] \\
&\qquad + o(n^{-1}).
\end{aligned}$$

In the proof of Lemma 2 we have seen that for nonzero integers k and for $w \in \mathbb{R}^p$ and uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0+u)} \exp(2\pi i k \gamma_n(Y, \alpha) + i w^T \tilde{Y}(n, \theta_0, \tau_0+u)) = o(n^0).$$

This implies that for $F\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ bounded, continuous, $F(0, x) = F(1, x)$ for all $x \in \mathbb{R}$, we have uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0+u)} F(\gamma_n(Y, \alpha), \tilde{Y}(n, \theta_0, \tau_0+u)) = \int_0^1 F(x, y)\, \varphi_{\Sigma_{11}}(y)\, dy\, dx + o(n^0).$$

Hence uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0+u)} h^*_m(\gamma_n(Y, \alpha))\, 1_{\{X=k_n(Y,\alpha)\}}$$

$$= n^{-1}\sigma^{-1}(\theta_0, \tau_0+u)\, \varphi(N_\alpha) \left\{ \sigma^{-1} N_\alpha \int_0^1 (x h^*_m(x) + \tfrac{1}{2} x^2 h^{*\prime}_m(x))\, dx \right.$$

$$\left. - \sigma^{-1} N_\alpha \int_0^1 x h^{*\prime}_m(x)\, G(y)\, \varphi_{\Sigma_{11}}(y)\, dy\, dx \right\} + o(n^{-1}).$$

Here, G is the polynomial of degree two or less satisfying

$$G(\tilde{y}(n, \theta_0, \tau_0+u)) = Q_1(v(y), \tilde{y}(n, \theta_0, \tau_0+u), \tau_0).$$

The term involving G has an unbounded smooth integrand. Here, a simple truncation argument is needed.

The assertion of Lemma 3 now follows from

$$\int_0^1 x h^{*\prime}_m(x)\, dx = 0$$

and

$$\int_0^1 (x h^*_m(x) + \tfrac{1}{2} x^2 h^{*\prime}_m(x))\, dx = 0.$$

Our next result is a modification of the Corollary in [2, p. 94]. Its proof is essentially the same as the one given in [2].

LEMMA 4. *For $y \in \mathbb{Z}^p$ let $\alpha_n(y) \in (0,1)$ be given and define $\gamma_n(y) \in [0,1)$ and $k_n(y) \in \mathbb{Z}$ by*

$$Q(n, y, \theta_0)\{w > k_n(y)\} + \gamma_n(y)\, Q(n, y, \theta_0)\{k_n(y)\} = \alpha_n(y),$$

and define

$$\begin{aligned} \varphi_n &= 1 && \text{if } X > k_n(Y) \\ &= \gamma_n(Y) && \text{if } X = k_n(Y) \\ &= 0 && \text{if } X < k_n(Y). \end{aligned}$$

Fix $\tau_0 \in T$ and assume that uniformly for τ with $|\tau - \tau_0| \leqslant n^{-1/2} \log n$ and $y \in A(n, (\theta_0, \tau))$

$$\alpha_n(y) - \alpha = o(n^{-1/3}).$$

If $\varphi_n \in \mathscr{T}_2^*(\tau_0)$, then uniformly for t in compact subsets of $\mathbb{R}$

$$P^n_{(\theta_0+tn^{-1/2},\tau_0)}(\varphi_n-\varphi_n^0)=I_2+o(n^{-1})$$

where I_2 is defined in Lemma 2 *of* [2, *p.* 92].

Remark 5. Theorem 6 in [2, p. 82] states that for $\tau_0 \in T_0$ we have for all $(\varphi_n)\in\mathscr{T}_2^*$ and $t>0$

$$P^n_{(\theta_0+tn^{-1/2},\tau_0)}\varphi_n \leqslant H^\alpha_{2n}(t,\theta_0,\tau_0)+o(n^{-1}).$$

From the proof given there we see that this inequality holds for all $(\varphi_n)\in\mathscr{T}_2^*(\tau_0)$.

To prove part (ii) of the theorem, fix $m\geqslant 1$ and define

$$\begin{aligned}\varphi_n^{**}&=1 && \text{if } X>k_n(Y,\alpha)\\ &=h_m(\gamma_n(Y,\alpha)) && \text{if } X=k_n(Y,\alpha)\\ &=0 && \text{if } X<k_n(Y,\alpha).\end{aligned}$$

Lemma 3 yields $(\varphi_n^{**})\in\mathscr{T}_2^*(\tau_0)$ for all $\tau_0\in T_0$. Lemma 4 implies that for all $\tau_0\in T$ with $M(\theta_0,\tau_0)\notin Q^p$ and $t>0$

$$P^n_{(\theta_0+tn^{-1/2},\tau_0)}(\varphi_n^{**}-\varphi_n^0)=n^{-1}t\sigma^{-1}\varphi(N_\alpha+t\sigma)\int_0^1 h_m^*(x)(x-\tfrac{1}{2})\,dx+o(n^{-1}).$$

The integral is positive since $h_m^*(x)<0$ for $x<\frac{1}{2}$, $h_m^*(x)>0$ for $x>\frac{1}{2}$.

To prove part (i) consider the family of test sequences $(\varphi_n^{(m)})\in\mathscr{T}_2^*(\tau_0)$ constructed with h_m. Let $m=m(n)$ tend to infinity with n such that the sequence (φ_n'), $\varphi_n'=\varphi_n^{m(n)}$, is still an element of $\mathscr{T}_2^*(\tau_0)$.

Let $\varphi_n^*=\varphi_n'$ if $|\tau_n-\tau_0|\leqslant 2n^{-1/2}\log n$, and $\varphi_n^*=\varphi_n^0$ elsewhere. Then $(\varphi_n^*)\in\mathscr{T}_2^*$. To see this let $\tau\in T$ be arbitrary. If $\tau\neq\tau_0$, then uniformly for $|u|\leqslant n^{1/2}\log n$

$$P^n_{(\theta_0,\tau+u)}\{|\tau_n-\tau_0|\leqslant 2n^{-1/2}\log n\}=o(n^{-1})$$

and therefore

$$P^n_{(\theta_0,\tau+u)}\varphi_n^*=P^n_{(\theta_0,\tau+u)}\varphi_n^0+o(n^{-1})=\alpha+o(n^{-1}).$$

If $\tau=\tau_0$, then uniformly for $|u|\leqslant n^{-1/2}\log n$

$$P^n_{(\theta_0,\tau_0+u)}\{|\tau_n-\tau_0-u|>n^{-1/2}\log n\}=o(n^{-1})$$

and therefore

$$P^n_{(\theta_0,\tau_0+u)}\{|\tau_n-\tau_0|>2n^{-1/2}\log n\}=o(n^{-1}).$$

This yields that uniformly for $|u| \leqslant n^{-1/2} \log n$

$$P^n_{(\theta_0, \tau_0 + u)} \varphi_n^* = P^n_{(\theta_0, \tau_0 + u)} \varphi'_n + o(n^{-1}) = \alpha + o(n^{-1}).$$

For $\tau \in T$ with $\tau \neq \tau_0$ we obtain from

$$P^n_{(\theta_0 + tn^{-1/2}, \tau)} \{ |\tau_n - \tau| > n^{-1/2} \log n \} = o(n^{-1})$$

that

$$P^n_{(\theta_0 + tn^{-1/2}, \tau)} (\varphi_n^* - \varphi_n^0) = o(n^{-1}).$$

Furthermore, for $t > 0$ we have

$$\begin{aligned} P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi_n^* &= P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi'_n + o(n^{-1}) \\ &= P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi_n^0 + n^{-1} t\sigma^{-1} \varphi(N_\alpha + t\sigma) \\ &\quad \times P^n_{(\theta_0 + tn^{-1/2}, \tau)} h^*_{m(n)}(\gamma_n(Y, \alpha))(\gamma_n(Y, \alpha) - \tfrac{1}{2}) + o(n^{-1}). \end{aligned}$$

For $0 < \varepsilon < \frac{1}{2}$ let $B(n, \alpha)$ be the set of all y for which $\gamma_n(y, \alpha)$ is an element of $[0, \varepsilon)$ or $(1 - \varepsilon, 1)$ or $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$. Then for n sufficiently large, $h_{m(n)}(\gamma_n(y, \alpha)) < \varepsilon$ if $y \notin B(n, \varepsilon)$ and $\gamma_n(y, \alpha) < \frac{1}{2}$, and $h_{m(n)}(\gamma_n(y, \alpha)) > 1 - \varepsilon$ if $y \notin B(n, \varepsilon)$ and $\gamma_n(y, \alpha) > \frac{1}{2}$. Hence, with

$$\begin{aligned} h_0^*(x) &= -x, & x &\leqslant \tfrac{1}{2} \\ &= 1 - x, & x &> \tfrac{1}{2} \end{aligned}$$

we have for n sufficiently large

$$|h^*_{m(n)}(\gamma_n(y, \alpha)) - h_0^*(\gamma_n(y, \alpha))| \leqslant \varepsilon + 2 1_{B(n, \varepsilon)}.$$

This implies that for n sufficiently large

$$\begin{aligned} &|P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} h^*_{m(n)}(\gamma_n(Y, \alpha))(\gamma_n(Y, \alpha) - \tfrac{1}{2}) \\ &\qquad - P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} h_0^*(\gamma_n(Y, \alpha))(\gamma_n(Y, \alpha) - \tfrac{1}{2})| \\ &\quad \leqslant \varepsilon + 2P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \{ Y \in B(n, \varepsilon) \}. \end{aligned}$$

Since under $P^n_{(\theta_0 + tn^{-1/2}, \tau_0)}$, $\gamma_n(Y, \alpha)$ is asymptotically uniformly distributed on $[0, 1]$, we have

$$\lim_n P^n_{(\theta_0 + tn^{-1/2}, \tau)} \{ Y \in B(n, \varepsilon) \} = 4\varepsilon$$

and

$$\lim_n P^n_{(\theta_0 + tn^{-1/2}, \tau)} h_0^*(\gamma_n(Y, \alpha))(\gamma_n(Y, \alpha) - \tfrac{1}{2}) = \tfrac{1}{24}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we obtain

$$\lim_n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} h^*_{m(n)}(\gamma_n(Y, \alpha))(\gamma_n(Y, \alpha) - \tfrac{1}{2}) = \tfrac{1}{24}$$

or

$$\lim_n P^n_{(\theta_0 + tn^{-1/2}, \tau_0)} \varphi^*_n = H^\alpha_{2n}(t, \theta_0, \tau_0) + o(n^{-1}).$$

According to our Remark 5, $H^\alpha_{2n}(t, \theta_0, \tau_0)$ is—up to errors of order $o(n^{-1})$—an upper bound for the power of tests in $\mathscr{T}^*_2(\tau_0)$. This proves property (b) of (φ^*_n). Property (c) of (φ^*_n) now follows from Theorem 2(iv) of [2, p. 78].

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